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Unsupervised multiphase segmentation: A recursive approach

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ABSTRACT

We propose an unsupervised multiphase segmentation algorithm based on Bresson et al.'s fast global minimization of Chan and Vese's two-phase piecewise constant segmentation model. The proposed algorithm recursively partitions a region into two subregions, starting from the largest scale. The segmentation process automatically terminates and detects when all the regions cannot be partitioned further. The number of regions is not given and can be arbitrary. Furthermore, this method provides a full hierarchical representation that gives a structure of a given image.

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1. Introduction

Image segmentation aims to partition an image domain into different regions in a meaningful way. Edge-based active contours methods [8,4,9] pose segmentation as an energy minimization problem and use edge detection functions that are based on local features to evolve contours towards object edges. Region-based active contours models incorporate both regions and edges to find a partition. Our proposed algorithm takes a region-based active contours approach because it is robust to noise and based on more global features. One of the early efforts towards region-based active contours was made by the Mumford and Shah segmentation model [11], which approximates a given image by a piecewise smooth image. However, the posed energy minimization problem is difficult to solve. Zhu and Yuille [17] use a family of Gaussian distributions to describe each region's data, i.e. mean and variance, and determine the boundaries of regions by competing with neighboring regions to best fit models at the largest possible areas. Their proposed energy minimization problem is also in general difficult to solve. Chan and Vese [6] proposed to solve a two-phase piecewise constant segmentation model, which is a variant of the Mumford and Shah model. The novelty of Chan and Vese is the use of the level set method to represent the evolving curve. The minimization is conveniently obtained by the gradient descent of the Euler-Lagrange equation of the energy functional.

The extension from the celebrated Chan and Vese's two-phase segmentation model to multiphase segmentation is not so natural, which is due to the nature of level sets. Several attempts have been made towards this extension. Vese and Chan [16] use n level set functions to represent 2^n regions because each level set function splits the image domain into two. This method implicitly represents the constraint of disjoint regions so no coupling forces are needed in order to constrain disjoint regions. However, when the number of regions is not a power of two, extra work has to be done. Chung and Vese [7] use only one level set function but with level lines other than the zero-level line to represent contours. This method can represent n regions and the constraint of disjoint regions is also implicitly dealt with. However, their model cannot deal with triple junctions and the authors suggest combining their method with the Vese and Chan model to overcome this problem. Lie et al. [10] introduce to segmentation a piecewise constant level set function to represent each phase with a constant value. The piecewise constant constraint on the level set function is solved by using the augmented Lagrangian. Their level set method does not require re-initialization that is necessary for the classical level set method. However, extra work, as described in their paper, is needed for noisy images. The segmentation methods described above do not require any training set but the number of regions or at least an upper bound has to be given.

Brox and Weickert [3] use one level set function for each region to represent Zhu and Yuille's model. Brox and Weickert propose using a coupled curve evolution to solve this multiphase segmentation model but assume the number of regions is known. They also propose to automatically find the number of regions by a

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coarse-to-fine strategy coupled with a hierarchical splitting. The authors apply a two-phase segmentation on a subregion, and if the Zhu and Yuille's energy functional is lowered, they continue this segmentation. This is repeated for all regions until the Zhu and Yuille's energy functional cannot be lowered. The number of phases obtained by this procedure is used in their multiphase segmentation model. Sandberg et al.'s piecewise constant segmentation model [13] automatically determines the number of regions and finds partitioning simultaneously. In the energy functional, they introduce a feature balancing term, the sum of all the inverse of region scales from each region, which is used to implicitly penalize the number of regions in addition to the total length of the boundaries. The *region scale* of a region is the quotient of its area and perimeter. Their model can be easily solved by a pixel-wise decision algorithm which implicitly deals with the disjoint constraint on all phases. This minimization method is very efficient but not robust to noise.

The followings are a few supervised multiphase segmentation models, although the focus of this work is an unsupervised method. Samson et al. [20] assume the number of regions is known, as well as the average intensity and variance in each region. Their model represents each region by a level set. Their proposed energy functional consists of three terms to enforce data fidelity, the regularity of the interface, and the constraints on no vacuum and overlapping of regions. Aujol and Chan [1,19] proposed a supervised classification framework for images with both textured and nontextured areas. The given image is first decomposed into a texture part and a geometric part. The data terms for the geometric part and texture part are treated by Samson et al.'s method and a wavelet-based level set evolution method [24]. Then, they use a logic framework to combine the results in a user definable way.

In this paper, we provide a spatial enclosure relationship between higher-level and lower-level regions so that one can analyze an image at a certain level of scale. *Scale* is related to contrast and region scale, and we use the definition of the TV scale in [18,21], which is defined as the time taken for a feature to disappear under the total variation flow. Tu and Zhu [15] consider segmentation a computing process rather than a vision task. The more one looks at an image, the more one sees. Therefore, segmentation results are not universal. We provide a "structure" of an image because that is how an image is usually interpreted. Following this idea, we propose to start from the coarsest partitioning and then refine each partitioning individually. Our proposed multiphase piecewise constant segmentation first applies the Chan and Vese model to partition an image domain into two and then recursively applies the Chan and Vese model in each partitioned region. This procedure gives a structure of an image implicitly utilizing the notion of "saliency" [14] that involves scale and intensity contrast in its determination. We additionally propose some stopping conditions to terminate the two-phase segmentation on the indicated region when it becomes meaningless to partition further. The stopping conditions use region scale and contrast to detect oversegmentation.

Fast Level Set Transform (FLST) [22,23] also provides a hierarchical representation of an image but is different from our algorithm. FLST uses a bottom-up region-growing algorithm to compute the family of lower (resp. upper) level sets with the increasing (resp. decreasing) inclusion property. With these inclusion properties, an image can be fully represented into a tree. Since their method is based on a local image feature, it is sensitive to noise; and therefore, a threshold is used to detect noise. Our proposed algorithm uses a top-down approach by recursively segmenting a region into two with a region-based model that is intrinsically robust to noise. In [22,23], an effective algorithm uses FLST and decomposes an image into a tree of shapes based on connected components of the level sets. A major advantage of this algorithm is due to the observation that with only intensity, the level set of an object may have undesired holes (closed level sets) inside their region; and their method is able to recognize the shape of the object without holes in it. Our algorithm does not assume this prior and on the other hand the segmentation process follows the notion of scale that is defined in the previous paragraph.

2. Two-phase piecewise constant segmentation on an indicated region

In this section, we first describe previous two-phase piecewise constant segmentation models and then present a natural extension to partition any given subregions that may be of arbitrary shapes. Let $f : \Omega \to [0, L]$ be the given grey-scale image. A two-phase piecewise constant version of the Mumford–Shah model [11] evolves a curve C towards the boundary between two regions and approximates f by two constants c_1 and c_2 inside the curve C and outside the curve C, respectively. The Chan and Vese model [6] is following energy minimization problem:

$$\inf_{C,c_1,c_2}\left\{E^1[C,c_1,c_2] = \int_C ds + \lambda\int_{inside(C)}(c_1 - f(x))^2\,dx + \lambda\int_{outside(C)}(c_2 - f(x))^2\,dx\right\}, \tag{1}$$

where the first term measures the total length of the curve C to penalize complicated interface between two regions and λ is a scalar parameter that controls the balance between regularization and data. This model can be represented in the following level-set formulation:

$$\inf_{\phi,c_1,c_2}\left\{E^1[\phi,c_1,c_2] = \int|\nabla H(\phi(x))|dx + \lambda\int H(\phi(x))(c_1 - f(x))^2\,dx + \lambda\int[1 - H(\phi(x))](c_2 - f(x))^2\,dx\right\}, \tag{2}$$

where H is the Heaviside function and ϕ is a level set function [12] such that $\phi > 0$ inside C and $\phi < 0$ outside C. The minimization of this level set formulation can be solved naturally by the standard PDE method [6] and allows topological changes of the curve. However, this model is not convex and thus a reasonable initialization is necessary to avoid getting stuck at undesired local minima. Chan et al. [5] proposed a convex model that solves (1).

Based on [5], Bresson et al. [2] proposed a fast global minimization of the Chan and Vese model. There are two major advantages of their algorithm. The first is that the initialization can be arbitrary. The second is that the solutions can be obtained much faster than the standard PDE method. Bresson et al.'s model is the following minimization problem:

$$\min_{u,0\leqslant v\leqslant 1,c_1,c_2}\left\{E^2_\Omega[u,v,c_1,c_2] = TV_\Omega(u) + \frac{1}{2\theta}\|u - v\|_{L^2(\Omega)} + \lambda\int_\Omega v(x)(c_1 - f(x))^2 + [1 - v(x)](c_2 - f(x))^2\,dx\right\}, \tag{3}$$

where θ is small enough so that u and v are significantly close to each other, λ is a parameter controlling the data fidelity term, and the total variation of u is defined in the following:

$$TV_\Omega(u) = \sup\left\{\int_\Omega u\,\mathrm{div}\,p\,dx \,|\, p\in C^1_c(\Omega;\mathbb{R}^2) : |p(x)|\leqslant 1,\ \ \forall x\in\Omega\right\}. \tag{4}$$

If $u^* = \mathrm{argmin}E^2_\Omega[u, v, c_1, c_2]$, the partition can be chosen to be, for instance, $\{u^* \geqslant 0.5\}$ and $\{u^* < 0.5\}$. Let $r(x, c_1, c_2) = (c_1 - f(x))^2 - (c_2 - f(x))^2$. The minimization is solved by alternating the following equations [2]:

$$c_1 = \frac{\int_\Omega f(x)v(x)dx}{\int_\Omega v(x)dx} \tag{5}$$

$$c_2 = \frac{\int_\Omega f(x)[1 - v(x)]dx}{\int_\Omega [1 - v(x)]dx} \tag{6}$$

$$p(x) = \frac{p(x) + \Delta t \nabla(\operatorname{div} p(x) - v(x)/\theta)}{1 + \Delta t |\nabla(\operatorname{div} p(x) - v(x)/\theta)|} \tag{7}$$

$$u = v - \theta \operatorname{div} p \tag{8}$$

$$v(x) = \min\{\max\{u(x) - \theta\lambda r(x, c_1, c_2), 0\}, 1\}, \tag{9}$$

where Δt is the time step. These equations are iterated until convergence. We generalize Bresson et al.'s algorithm described above for partitioning a given region $S \subseteq \Omega$ that may have arbitrary shapes, in the following:

$$\min_{u, 0 \leqslant v \leqslant 1, c_1, c_2} \left\{ E_S^2[u, v, c_1, c_2] = TV_S(u) + \frac{1}{2\theta}\|u - v\|_{L^2(S)} + \lambda \int_S v(x)(c_1 - f(x))^2 + [1 - v(x)](c_2 - f(x))^2\, dx \right\}, \tag{10}$$

The minimization equations are the same as Eqs. (5)–(9), except the solutions are restricted on the indicated region S. The discretization of div and ∇ that satisfy the definition of TV norm in Eq. (4) can be defined in the following ways to satisfy the definition of the TV norm in Eq. (4). Write $p = (p^1, p^2)$. For all (i, j) such that $\chi_S(i, j) = 1$,

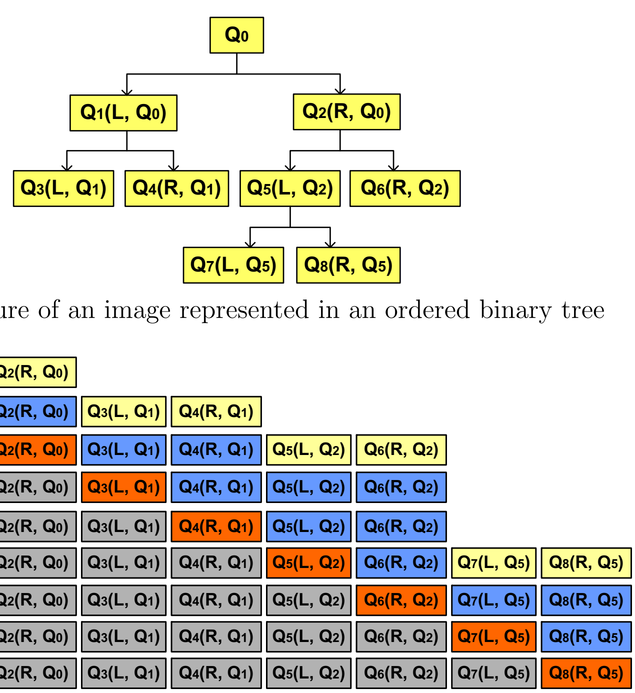
$$(\nabla u)^1_{i,j} = \chi_S(i+1, j)(u_{i+1,j} - u_{i,j}) \tag{11}$$

$$(\nabla u)^2_{i,j} = \chi_S(i, j+1)(u_{i,j+1} - u_{i,j}) \tag{12}$$

$$(\operatorname{div} p)_{i,j} = \chi_S(i+1, j)p^1_{i,j} - \chi_S(i-1, j)p^1_{i-1,j} + \chi_S(i, j+1)p^1_{i,j} - \chi_S(i, j-1)p^1_{i,j-1}. \tag{13}$$

3. Hierarchical representation of an image

In this section, we present our proposed segmentation algorithm which provides a full hierarchical representation of the structure of a given image. The proposed algorithm recursively applies Eq. (10) to split a partitioned region into two and generates an ordered binary tree to represent the structure of the image. Fig. 1a is an example of the process of the proposed recursive segmentation. Initially, the two-phase piecewise constant segmentation is applied on the entire image domain so the root node, Q_0, represents the entire image domain. The partitioned regions of Q_0 are stored in $Q_1(L, Q_0)$ and $Q_2(R, Q_0)$ where the second place, Q_0, indicates the parent node, and L and R represent left child node and right child node, respectively. Since the initialization of the two-phase segmentation (10) can be arbitrary, we may conveniently choose the image itself (normalized to range from 0 to 1) as initialization for v in (10). The segmented regions are $\{u \geqslant th\}$ and $\{u < th\}$, where th is the intensity mean of the current target region (or the union of both segmented regions). The left child node represents region $\{u \geqslant th\}$ and the right child represents region $\{u < th\}$. Since the minimization is a gradient descent, by our initialization, the intensity mean of the left child node is always higher than that of the right child node. In this way, the order of intensity means of the partitioned regions are preserved and we obtain an ordered binary tree. Next, we apply the two-phase segmentation on the region in $Q_1(L, Q_0)$ and store the segmented regions into $Q_3(L, Q_1)$ and $Q_4(R, Q_1)$. Then, we continue this method and proceed segmentation on $Q_2(L, Q_0)$, which is split into $Q_5(L, Q_2)$ and $Q_6(R, Q_2)$. This process gives us a hierarchical representation of the image structure. However, this process becomes meaningless when the scale of structure becomes too small. Therefore, we propose to terminate segmentation of a region if one of the following three conditions is satisfied. The first is when the evolved curve disappears, which happens naturally if the current region has homogeneous intensity. The second stopping condition uses region scale to prevent over-segmentation. The third is when the two approximated constants are so similar that it becomes meaningless to partition the current region. In this fashion, a region that satisfies any of the stopping



(a) Structure of an image represented in an ordered binary tree

(b) Proposed algorithm

Fig. 1. An example of the proposed recursive two-phase segmentation. (a) Ordered binary tree that represents the structure of an image. (b) Process, from top to bottom, of the recursive segmentation in which each partitioned region is stored in a queue. The bottom row is the final tree structure in the queue representation.

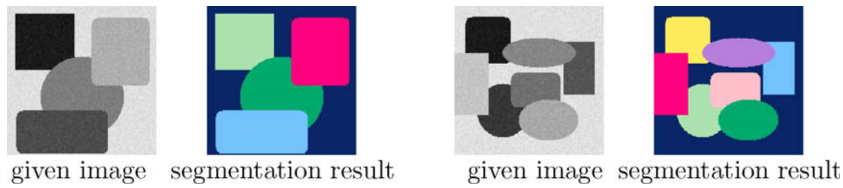


Fig. 2. The proposed algorithm automatically detects each region in the given images and junctions are preserved. Each color represents a phase of the segmentation. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

condition above becomes a leaf of the ordered binary tree. In the example of Fig. 1, region $Q_3(L, Q_1)$ meets one of the stopping criteria and thus has no child nodes. Similarly, region $Q_4(R, Q_1)$ has no child nodes.

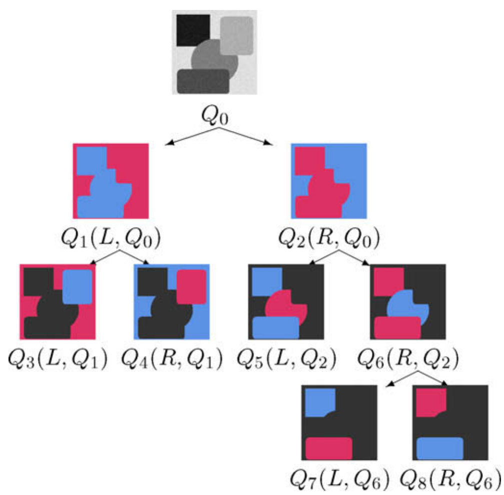


Fig. 3. Image structure by the proposed segmentation. Red: extracted region, black: irrelevant region, blue: complement of red in relevant region. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

To implement the recursive segmentation described above, we use a queue to store the segmented regions. Fig. 1b shows the procedure, from top to bottom, of the implementation of the example in Fig. 1a. The first row shows that the current node is Q_0 and the partitioned regions are stored into $Q_1(L, Q_0)$ and $Q_2(R, Q_0)$. The subindex of Q is the position of entity in the queue. The second row shows that the current node is $Q_1(L, Q_0)$ and the partitioned regions are stored into $Q_3(L, Q_1)$ and $Q_4(R, Q_1)$. The third row shows that the current node is $Q_2(R, Q_0)$ and the partitioned regions are stored into $Q_5(L, Q_2)$ and $Q_6(R, Q_2)$. The forth row shows that the current node is $Q_3(L, Q_1)$ and does not need to be partitioned. This process is continued until there is no more node stored in the queue.

The general formulation of our proposed framework, which is explained above in an example, is shown in Algorithm 1. Initially, the region to be segmented is the entire image domain and is represented by a characteristic function χ_Ω. 'Phase' is a matrix of the size of the image that records leaf regions and j is the label of each phase; n keeps track of current entity in queue to be segmented, and N is the position of the entity in queue where a newly segmented region is to be stored. The function 'two-phase piecewise constant segmentation' uses Eq. (10) and returns two characteristic functions χ_1 and χ_2 that represent the segmented regions of the current region. It additionally returns a true/false variable 'split' according to the stopping conditions. If one of the stopping conditions is satisfied, 'split' is false; otherwise, 'split' is true.

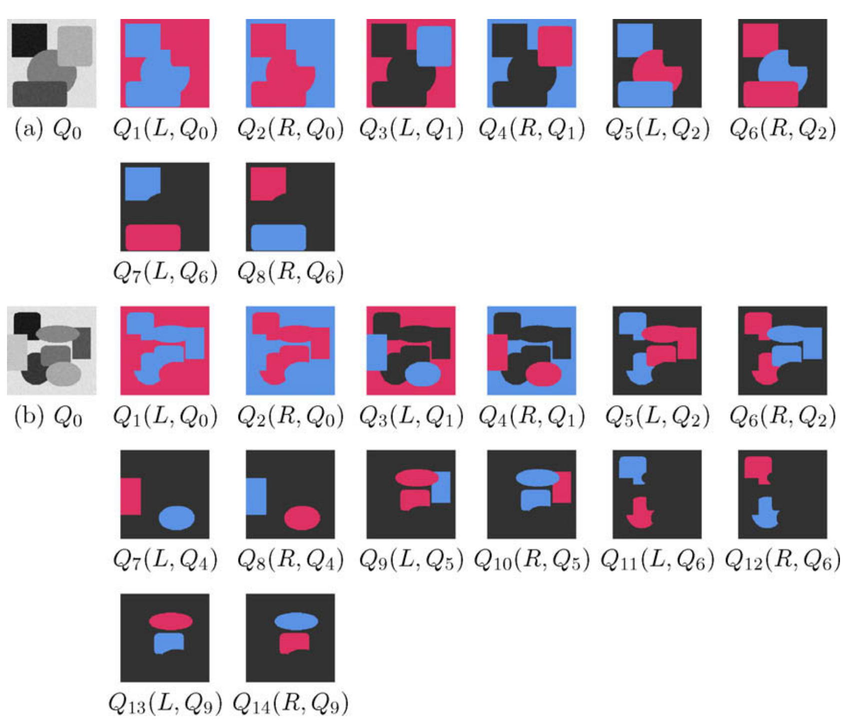


Fig. 4. Tree structures of image (a) and image (b) in the queue representation.

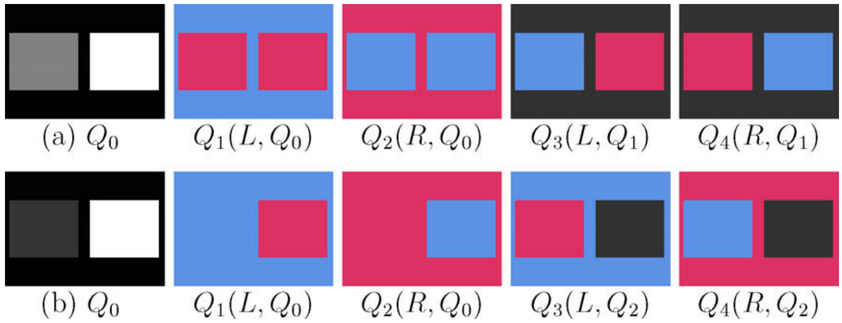


Fig. 5. (a) Segmentation starts from the largest region scale. In (b), the right rectangle has a higher contrast and its region scale is also large enough, so segmentation first separates it from the rest.

Algorithm 1 Recursive two-phase segmentation

Given image $f : \Omega \to [0, L]$
$Q_0 \Leftarrow \chi_\Omega$
Phase $\Leftarrow \vec{0}$
$n \Leftarrow 0$
$N \Leftarrow 0$
$j \Leftarrow 0$
while Q_n exists **do**
  $[\chi_1, \chi_2$,split]=two-phase piecewise constant segmentation(f, Q_n)
  if split=true bf then
    $Q_{N+1}(L, Q_n) \Leftarrow \chi_1$
    $Q_{N+2}(R, Q_n) \Leftarrow \chi_2$
    $N \Leftarrow N + 2$
  else
    Phase$\Leftarrow$Phase+$j * Q_n$
    $j \Leftarrow j + 1$
endif
$n \Leftarrow n + 1$
end while

4. Experimental results

Fig. 2 shows experiments on synthetic images with five and eight regions, respectively. The proposed algorithm automatically detects each region of the images. The segmentation results are shown in different colors to represent each phase of the segmentation. As can be seen, the recursive approach implicitly deals with junctions. Fig. 3 shows the ordered binary tree structure of the image. The red region represents the extracted region, black represents the irrelevant region, and blue represents the complement of the red in the relevant region. Fig. 4 shows the binary tree structures of image (a) and image (b) in the queue representation. Due to space limitations, other results of image structures are only shown in the queue representation.

In Fig. 5, (a) shows that the proposed segmentation starts from the coarsest scale. The two rectangles are first separated from the background and then separated from each other. In (b), the contrast of the right rectangle is high and also the region scale of it is high enough. Therefore, the right rectangle is first separated from the union of the left rectangle and background, and then the left rectangle and background are separated. This is the case when the contrast is high enough to have greater influence on

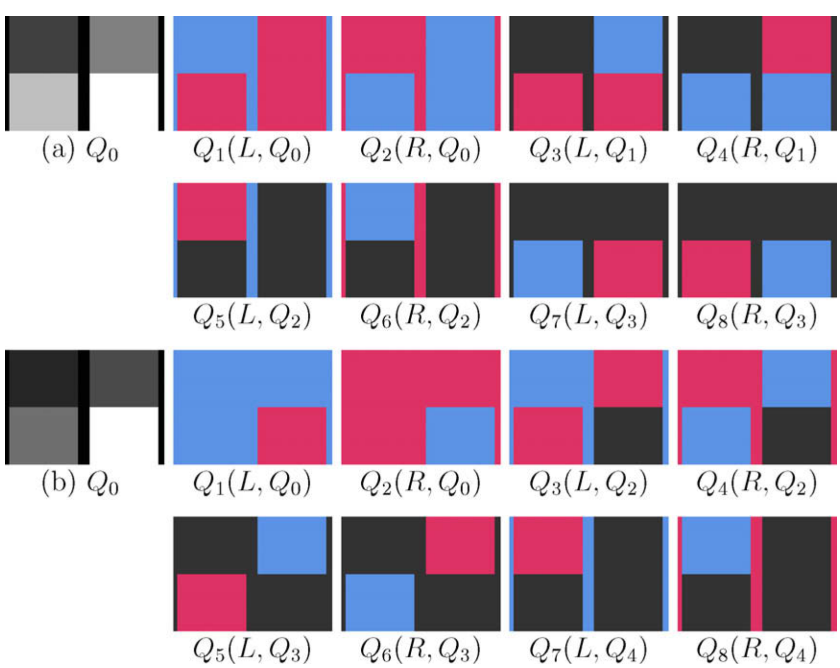


Fig. 6. The proposed segmentation starts from the largest region scale and highest contrast. In (a), the contrasts from the background to the top-left rectangle, from the top-left to the top-right, from the top-right to the bottom-left, and from the bottom-left to the bottom-right are all the same. All five regions have the same area but the perimeter of the background is larger than those of the rectangles. Therefore, the first segmentation separates the rectangles with the lowest intensity and the background from the rest. (b) The bottom-right rectangle has the highest contrast and is extracted first.

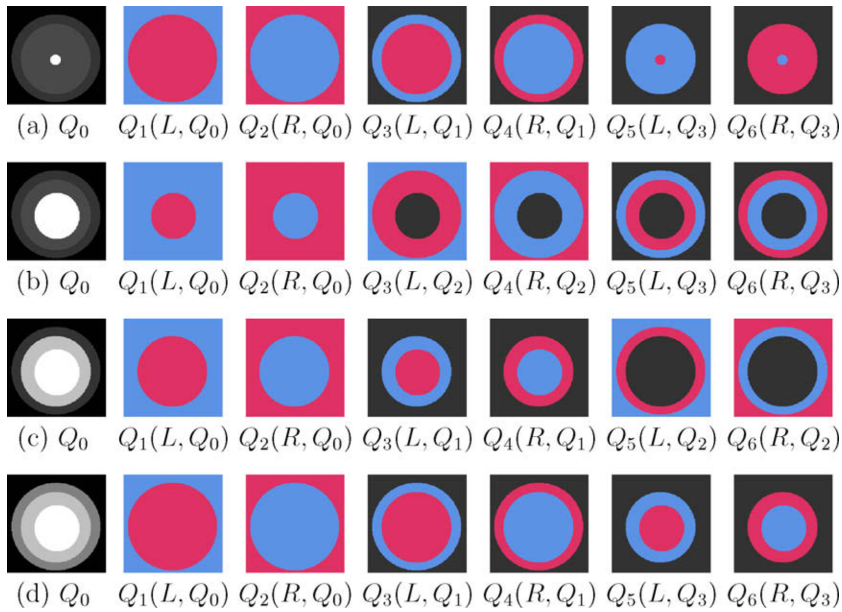


Fig. 7. The segmentation is from the largest region scale, in (a) and (d), but also depends on the contrast when region scale is large enough, in (b) and (c).

the segmentation result than the region scale. Fig. 6 also shows that recursive segmentation competes between the region scale and contrast in a reasonable manner. In (a), the contrasts from the background to the top-left rectangle, from the top-left to the top-right, from the top-right to the bottom-left, and from the bottom-left to the bottom-right are all the same. All five regions have the same area but the perimeter of the background is larger than those of the rectangles. Therefore, as expected according to region scale, the first segmentation separates the rectangle with lowest intensity and the background from the rest of the rectangles. In (b), the bottom-right rectangle has the highest contrast and its region scale is large enough. Therefore, the first segmentation extracts it from other regions. Fig. 7 shows that the order of segmentation is from the largest region scale, as in (a) and (d), and also depends on the contrast, as in (b) and (c). In (a), the center circle has a very high contrast but its region scale is small. Therefore, all the circles as a unit are separated from the background in the first segmentation step. In (b–d), all high-contrasted circles' scales are large enough and thus the segmentation is performed according to the contrast. All results show that recursive segmentation competes region scale and contrast in an intuitive way.

Fig. 8 shows the segmentation results of some real images. The proposed algorithm automatically detects each region of the given images. The segmentation results are obtained by applying the proposed algorithm on the respective grey-scale images of the color images shown in the figure. The proposed algorithm can be naturally applied to vector images such as color images. Fig. 9 shows that the images are first roughly partitioned into two regions of similar sizes and then partitioned at finer scale as the tree level goes down. In (a), the sky and far mountains are separated from the near mountains and trees. Then, the sky and the far mountains are separated and the near mountains and trees are separated. Finally, different layers of the near mountains are separated. In (b), the sky and far mountains are separated from near mountains and rocks. Then, the sky and far mountains are separated and the near mountains and rocks are separated. In (c), the trees and sky are separated first. Then, the sky is partitioned into two because its intensity is not homogeneous. Finally, the moon is separated from the sky. In (d), the sky and the buildings are separated first, except for the roof because its intensity is similar to that of the sky. Then, the roof is separated from the sky.

Fig. 10 shows experimental results of other natural images. The first row shows a given image, its final segmentation, and the associated binary tree structure in a queue representation. Initially, the woman is separated from the background. Then, her skin and clothes are segmented into different regions. The third and fourth rows show the given images and their segmentation results. Due to space limitation, the binary tree structures are not shown. One can

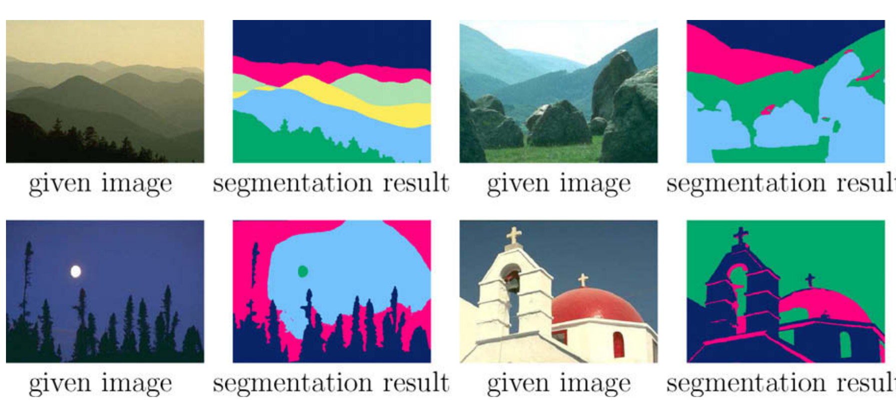


Fig. 8. The proposed algorithm automatically detects each region in the given images. The segmentation results are obtained by experimenting on the respective grey-scale images of the color images shown here. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

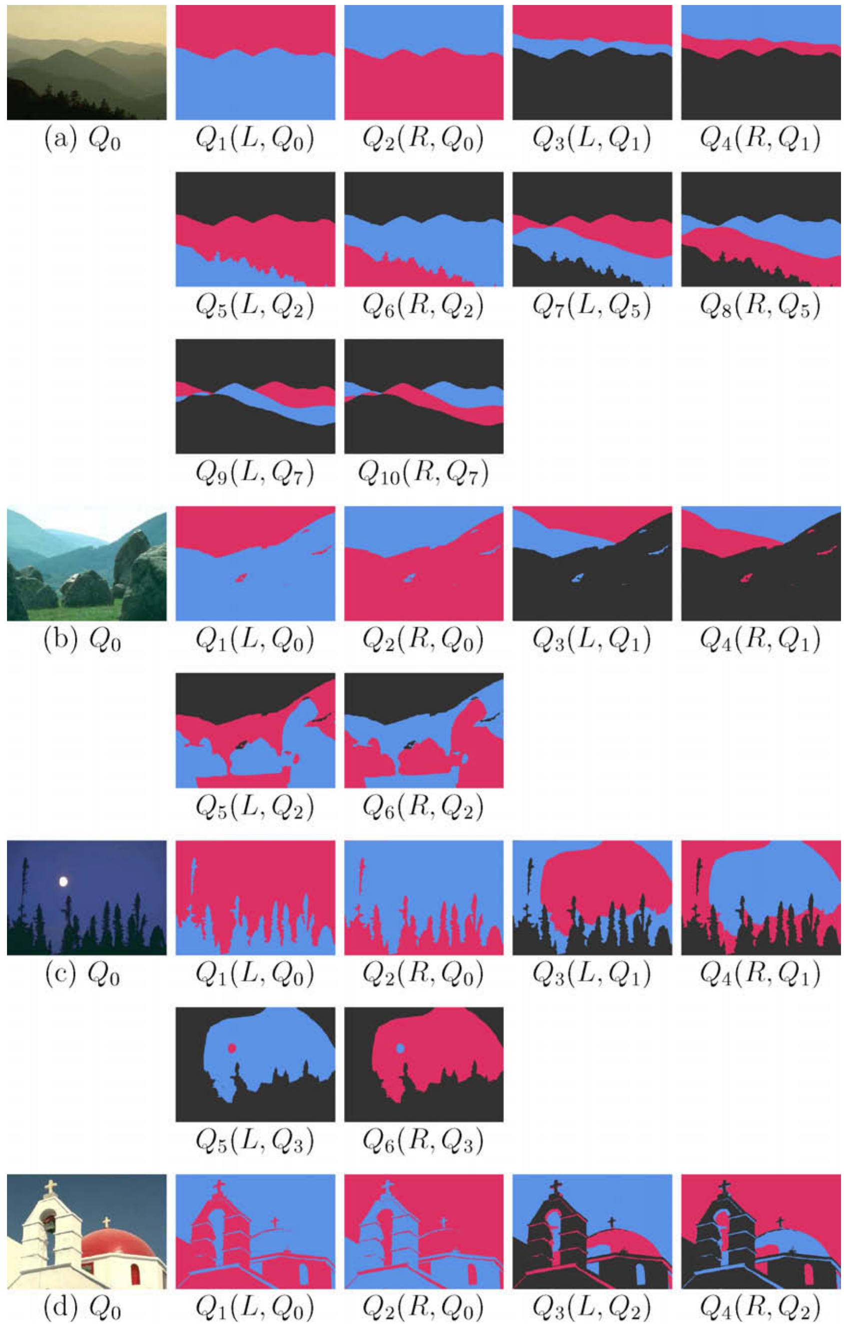


Fig. 9. The structures of the images in the queue representation. These images are first roughly partitioned into two regions of similar sizes and then partitioned at finer scale as the tree level goes down. (a) First the sky and far mountains are separated from the near mountains and trees. (b) First the sky and far mountains are separated from near mountains and rocks. (c) The trees and sky are separated first. (d) The sky and the buildings are separated first, except for the roof because its intensity is close to that of the sky. The experiments are on the respective grey-scale images of the original color images shown here. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

see that the proposed algorithm is able to accomplish multiphase segmentation of a variety of natural images.

5. Conclusions and discussion

In this paper, we proposed a segmentation algorithm that provides the structure of an image from the largest scale. The proposed algorithm recursively applies Bresson et al.'s two-phase piecewise constant segmentation on the partitioned region. We defined an ordered binary tree to represent this process to show the structure of an image. This ordered binary tree is implemented by storing each node as an entity in a queue. We proposed three stopping conditions to detect when segmenting a region further is trivial. This can be used to give a multiphase segmentation result, in which the leaf of the tree represents each phase of the multiphase segmentation. Given the stopping conditions, the number of phases can be unknown and can be arbitrary. Additionally, junctions of contours are implicitly dealt with by the recursive segmentation method. In our framework, we have employed a level set segmentation algorithm that provides the ability to maintain the identity of a region as it splits and merges into separate connected component. It may be useful to maintain both the separate identity of each connected component, as well as the knowledge that they are linked - namely they are individual connected components of

Fig. 10. The first row, from left to right, shows the given image, a final segmentation, and the associated binary tree structure in the queue representation. The second and third rows are several segmentation results of the given images.

the same region. This property can be integrated into our framework by adding a layer of connected component analysis, that can be efficiently implemented using standard region-filling algorithms.We have shown several experiments on synthetic images to see that the proposed segmentation follows the order of region scale and contrast in a reasonable manner. Experimental results on real images show that the obtained image structures are consistently intuitive.

Acknowledgements

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References

[1] J.F. Aujol, G. Gilboa, T.F. Chan, S. Osher, Structure–texture image decomposition—modeling, algorithms, and parameter selection, IJCV 67 (1) (2006) 151–167.
[2] X. Bresson, S. Esedognlu, P. Vandergheynst, J.-P. Thiran, S. Osher, Fast global minimization of the active contour/snake model, Journal of Mathematical Imaging and Vision 28 (2) (2007) 151–167.
[3] T. Brox, J. Weickert, Level set segmentation with multiple regions, IEEE Transactions on Image Processing 15 (10) (2006) 3213–3218.
[4] V. Caselles, R. Kimmel, G. Sapiro, Geodesic active contours, International Journel of Computer Vision 22 (1) (1997) 61–79.
[5] T.F. Chan, S. Esedoglu, M. Nikolova, Algorithms for finding global minimizers of image segmentation and denoising models, SIAM Journal on Applied Mathematics 66 (5) (2006) 1632–1648.
[6] T.F. Chan, L.A. Vese, Active contours without edges, IEEE Transactions on Image Processing 10 (2) (2001) 266–277.
[7] J.T. Chung, L.A. Vese, Image segmentation using a multilayer level-set approach, Technical Report CAM03-53.
[8] M. Kass, A. Witkin, D. Terzopoulos, Snakes: active contour models, International Journal of Computer Vision (1987) 321–331.
[9] S. Kichenassamy, A. Kumar, P. Olver, A. Tannenbaum, A. Yezzi, Conformal curvature flows: form phase transitions to active vision, Archive for Rational Mechanics and Analysis 134 (1996) 275–301.
[10] J. Lie, M. Lysaker, X.-C. Tai, A variant of the level set method and applications to image segmentation, AMS Mathematics of Computation 75 (2006) 1155–1174.
[11] D. Mumford, J. Shah, Optimal approximations by piecewise smooth functions and associated variational problems, Communications on Pure and Applied Mathematics 42 (1989) 577–685.
[12] S. Osher, J.A. Sethian, Front propagating with curvature-dependent speed: algorithms based on Hamilton–Jacobi formulation, Journal of Computational Physics 79 (1988) 12–49.
[13] B. Sandberg, S.H. Kang, T.F. Chan, Unsupervised feature balancing multi-phase segmentation, Technical Report CAM08-02.
[14] A. Shashua, S. Ullman, Structural saliency: the detection of globally salient structures using a locally connected network, International Conference on Computer Vision 5 (8) (1988) 321–327.
[15] Z. Tu, S.C. Zhu, Image segmentation by data-driven markov chain monte carlo, IEEE Transactions on Pattern Analysis and Machine Intelligence 24 (5) (2002) 657–673.
[16] L.A. Vese, T.F. Chan, A multiphase level set framework for image segmentation using the mumford and shah model, International Journal of Compter Vision 50 (3) (2002) 271–293.
[17] S.C. Zhu, A. Yuille, Region competition: unifying snakes, region growing, and bayes/mdl for multiband image segmentation, IEEE Transactions on Pattern Analysis and Machine Intelligence 18 (1996) 884–900.
[18] D.M. Strong, J.-F. Aujol, T.F. Chan, Scale recognition, regularization parameter selection, and Meyer's G norm in total variation regularization, SIAM Multiscale Modeling and Simulation 5 (1) (2006) 273–303.
[19] J.-F. Aujol, T.F. Chan, Combining geometrical and textured information to perform image classification, Journal of Visual Communication and Image Representation 17 (2006) 1004–1023.
[20] C. Samson, L. Blanc-Feraud, G. Aubert, J. Zerubia, A level set model for image classification, International Journal of Computer Vision 40 (3) (2002) 187–197.

[21] A. Petrovic, O.D. Escoda, P. Vandergheynst, Multiresolution segmentation of natural images: from linear to nonlinear scale-space representations, IEEE Transactions on Image Processing 13 (8) (2004) 1104–1114.
[22] P. Monasse, F. Guichard, Fast computation of a contrast-invariant image representation, IEEE Transactions on Image Process 9 (5) (2000) 860–872.
[23] V. Caselles, P. Monasse, Geometric description of topographic maps and applications to image processing, CMLA Preprint 2008-24.
[24] J.-F. Aujol, G. Aubert, L. Blanc-Feraud, Wavelet-based level set evolution for classification of textured images, IEEE Transactions on Image Processing 12 (12) (2003) 1634–1641.